

Bounded perturbations of two-dimensional diffusion processes with nonlocal conditions near the boundary

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Abstract

We study the existence of Feller semigroups arising in the theory of multidimensional diffusion processes. We study bounded perturbations of elliptic operators with boundary conditions containing an integral over the closure of the domain with respect to a nonnegative Borel measure without assuming that the measure is small. We state sufficient conditions on the measure guaranteeing that the corresponding nonlocal operator is the generator of a Feller semigroup.

1 Introduction and Preliminaries

In [2, 3], Feller investigated a general form of a generator of a strongly continuous contractive non-negative semigroup of operators acting between the spaces of continuous functions on an interval, a half-line, or the whole line. Such a semigroup corresponds to the one-dimensional diffusion process and is now called the Feller semigroup. In the multidimensional case, the general form of a generator of a Feller semigroup has been obtained by Ventsel [14]. Under some regularity assumptions concerning the Markov process, he proved that the generator of the corresponding Feller semigroup is an elliptic differential operator of second order (possibly with degeneration) whose domain of definition consists of continuous (once or twice continuously differentiable, depending on the process) functions satisfying nonlocal conditions which involve an integral of a function over the closure of the region with respect to a nonnegative Borel measure $\mu(y, d\eta)$. The inverse question remains open: given an elliptic integro-differential operator whose domain of definition is described by nonlocal boundary conditions, whether or not this operator (or its closure) is a generator of a Feller semigroup.

One distinguishes two classes of nonlocal boundary conditions: the so-called *transversal* and *non-transversal* ones. The order of nonlocal terms is less than the order of local terms in the transversal case, and these orders coincide in the nontransversal case (see, e.g., [13] for details and probabilistic interpretation). The transversal case was studied in [9, 1, 12, 13, 8, 6]. The more difficult non-transversal nonlocal conditions are dealt with in [10, 11, 5, 6].

It was assumed in [10, 11] that the coefficients at nonlocal terms decrease as the argument tends to the boundary. In [5, 6], the authors considered nonlocal conditions with the coefficients that are less than one. This allowed them to regard (after reduction to the boundary) the nonlocal problem as a perturbation of the “local” Dirichlet problem.

In this paper, we consider nontransversal nonlocal conditions on the boundary of a plane domain G , admitting “limit case” where the measure $\mu(y, \overline{G})$, after some normalization, may equal one (it cannot be greater than one [14]). We assume that if the support of the measure $\mu(y, d\eta)$ is “close” to the point y for some $y \in \partial G$ and $\mu(y, \overline{G}) = 1$, then the measure $\mu(y, d\eta)$ is atomic.

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Based on the Hille–Iosida theorem and on the solvability of elliptic equations with nonlocal terms supported near the boundary [7], we provide a class of Borel measures $\mu(y, d\eta)$ for which the corresponding nonlocal operator is a generator of a Feller semigroup.

In the conclusion of this section, we remind the notion of a Feller semigroup and its generator and formulate a version of the Hille–Iosida theorem adapted for our purposes.

Let $G \subset \mathbb{R}^2$ be a bounded domain with piecewise smooth boundary ∂G , and let X be a closed subspace in $C(\overline{G})$ containing at least one nontrivial nonnegative function.

A strongly continuous semigroup of operators $\mathbf{T}_t : X \rightarrow X$ is called a *Feller semigroup on X* if it satisfies the following conditions: 1. $\|\mathbf{T}_t\| \leq 1$, $t \geq 0$; 2. $\mathbf{T}_t u \geq 0$ for all $t \geq 0$ and $u \in X$, $u \geq 0$.

A linear operator $\mathbf{P} : D(\mathbf{P}) \subset X \rightarrow X$ is called the (*infinitesimal*) *generator* of a strongly continuous semigroup $\{\mathbf{T}_t\}$ if $\mathbf{P}u = \lim_{t \rightarrow +0} (\mathbf{T}_t u - u)/t$, $D(\mathbf{P}) = \{u \in X : \text{the limit exists in } X\}$.

Theorem 1.1 (the Hille–Iosida theorem, see Theorem 9.3.1 in [12]). *1. Let $\mathbf{P} : D(\mathbf{P}) \subset X \rightarrow X$ be a generator of a Feller semigroup on X . Then the following assertions are true.*

- (a) *The domain $D(\mathbf{P})$ is dense in X .*
- (b) *For each $q > 0$ the operator $q\mathbf{I} - \mathbf{P}$ has the bounded inverse $(q\mathbf{I} - \mathbf{P})^{-1} : X \rightarrow X$ and $\|(q\mathbf{I} - \mathbf{P})^{-1}\| \leq 1/q$.*
- (c) *The operator $(q\mathbf{I} - \mathbf{P})^{-1} : X \rightarrow X$, $q > 0$, is nonnegative.*

- 2. Conversely, if \mathbf{P} is a linear operator from X to X satisfying condition (a) and there is a constant $q_0 \geq 0$ such that conditions (b) and (c) hold for $q > q_0$, then \mathbf{P} is the generator of a certain Feller semigroup on X , which is uniquely determined by \mathbf{P} .*

2 Nonlocal Conditions near the Conjugation Points

Consider a set $\mathcal{K} \subset \partial G$ consisting of finitely many points. Let $\partial G \setminus \mathcal{K} = \bigcup_{i=1}^N \Gamma_i$, where Γ_i are open (in the topology of ∂G) C^∞ curves. Assume that the domain G is a plane angle in some neighborhood of each point $g \in \mathcal{K}$.

For an integer $k \geq 0$, denote by $W_2^k(G)$ the usual Sobolev space. Denote by $W_{2,\text{loc}}^k(G)$ ($k \geq 0$ is an integer) the set of functions u such that $u \in W_2^k(G')$ for any domain $G', \overline{G'} \subset G$.

Consider the differential operator

$$P_0 u = \sum_{j,k=1}^2 p_{jk}(y) u_{y_j y_k}(y) + \sum_{j=1}^2 p_j(y) u_{y_j}(y) + p_0(y) u(y),$$

where $p_{jk}, p_j \in C^\infty(\mathbb{R}^2)$ are real-valued functions and $p_{jk} = p_{kj}$, $j, k = 1, 2$.

Condition 2.1. *1. There is a constant $c > 0$ such that $\sum_{j,k=1}^2 p_{jk}(y) \xi_j \xi_k \geq c|\xi|^2$ for $y \in \overline{G}$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. 2. $p_0(y) \leq 0$ for $y \in \overline{G}$.*

In the sequel, we will use the following version of the well-known maximum principle.

Maximum Principle 2.1 (see Theorem 9.6 in [4]). *Let $D \subset \mathbb{R}^2$ be a bounded or unbounded domain, and let Condition 2.1 hold with G replaced by D . If a function $u \in C(D)$ achieves its positive maximum at a point $y^0 \in D$ and¹ $P_0 u \in C(D)$, then $P_0 u(y^0) \leq 0$.*

¹Here and below the operator P_0 acts in the sense of distributions.

Introduce the operators corresponding to nonlocal terms supported near the set \mathcal{K} . For any set \mathcal{M} , we denote its ε -neighborhood by $\mathcal{O}_\varepsilon(\mathcal{M})$. Let Ω_{is} ($i = 1, \dots, N$; $s = 1, \dots, S_i$) be C^∞ diffeomorphisms taking some neighborhood \mathcal{O}_i of the curve $\overline{\Gamma_i} \cap \mathcal{O}_\varepsilon(\mathcal{K})$ to the set $\Omega_{is}(\mathcal{O}_i)$ in such a way that $\Omega_{is}(\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})) \subset G$ and $\Omega_{is}(g) \in \mathcal{K}$ for $g \in \overline{\Gamma_i} \cap \mathcal{K}$. Thus, the transformations Ω_{is} take the curves $\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})$ strictly inside the domain G and the set of their end points $\overline{\Gamma_i} \cap \mathcal{K}$ to itself.

Let us specify the structure of the transformations Ω_{is} near the set \mathcal{K} . Denote by Ω_{is}^{+1} the transformation $\Omega_{is} : \mathcal{O}_i \rightarrow \Omega_{is}(\mathcal{O}_i)$ and by $\Omega_{is}^{-1} : \Omega_{is}(\mathcal{O}_i) \rightarrow \mathcal{O}_i$ the inverse transformation. The set of points $\Omega_{i_1 s_1}^{\pm 1}(\dots \Omega_{i_q s_q}^{\pm 1}(g)) \in \mathcal{K}$ ($1 \leq s_j \leq S_{i_j}$, $j = 1, \dots, q$) is said to be an *orbit* of the point $g \in \mathcal{K}$. In other words, the orbit of a point g is formed by the points (of the set \mathcal{K}) that can be obtained by consecutively applying the transformations $\Omega_{i_j s_j}^{\pm 1}$ to the point g . The set \mathcal{K} consists of finitely many disjoint orbits, which we denote by \mathcal{K}_ν .

Take a sufficiently small number $\varepsilon > 0$ such that there exist neighborhoods $\mathcal{O}_{\varepsilon_1}(g_j)$, $\mathcal{O}_{\varepsilon_1}(g_j) \supset \mathcal{O}_\varepsilon(g_j)$, satisfying the following conditions: 1. the domain G is a plane angle in the neighborhood $\mathcal{O}_{\varepsilon_1}(g_j)$; 2. $\overline{\mathcal{O}_{\varepsilon_1}(g)} \cap \overline{\mathcal{O}_{\varepsilon_1}(h)} = \emptyset$ for any $g, h \in \mathcal{K}$, $g \neq h$; 3. if $g_j \in \overline{\Gamma_i}$ and $\Omega_{is}(g_j) = g_k$, then $\mathcal{O}_\varepsilon(g_j) \subset \mathcal{O}_i$ and $\Omega_{is}(\mathcal{O}_\varepsilon(g_j)) \subset \mathcal{O}_{\varepsilon_1}(g_k)$.

For each point $g_j \in \overline{\Gamma_i} \cap \mathcal{K}_\nu$, we fix a linear transformation $Y_j : y \mapsto y'(g_j)$ (the composition of the shift by the vector $-\overrightarrow{\mathcal{O}g_j}$ and rotation) mapping the point g_j to the origin in such a way that $Y_j(\mathcal{O}_{\varepsilon_1}(g_j)) = \mathcal{O}_{\varepsilon_1}(0)$, $Y_j(G \cap \mathcal{O}_{\varepsilon_1}(g_j)) = K_j \cap \mathcal{O}_{\varepsilon_1}(0)$, $Y_j(\Gamma_i \cap \mathcal{O}_{\varepsilon_1}(g_j)) = \gamma_{j\sigma} \cap \mathcal{O}_{\varepsilon_1}(0)$ ($\sigma = 1$ or 2), where K_j is a plane angle of nonzero opening and $\gamma_{j\sigma}$ its sides.

Condition 2.2. Let $g_j \in \overline{\Gamma_i} \cap \mathcal{K}_\nu$ and $\Omega_{is}(g_j) = g_k \in \mathcal{K}_\nu$; then the transformation $Y_k \circ \Omega_{is} \circ Y_j^{-1} : \mathcal{O}_{\varepsilon_1}(0) \rightarrow \mathcal{O}_{\varepsilon_1}(0)$ is the composition of rotation and homothety centered at the origin.

Introduce the nonlocal operators \mathbf{B}_i by the formulas

$$\mathbf{B}_i u = \sum_{s=1}^{S_i} b_{is}(y) u(\Omega_{is}(y)), \quad y \in \Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K}), \quad \mathbf{B}_i u = 0, \quad y \in \Gamma_i \setminus \mathcal{O}_\varepsilon(\mathcal{K}), \quad (2.1)$$

where $b_{is} \in C^\infty(\mathbb{R}^2)$ are real-valued functions, $\text{supp } b_{is} \subset \mathcal{O}_\varepsilon(\mathcal{K})$.

Condition 2.3. 1. $b_{is}(y) \geq 0$, $\sum_{s=1}^{S_i} b_{is}(y) \leq 1$, $y \in \overline{\Gamma_i}$;
2. $\sum_{s=1}^{S_i} b_{is}(g) + \sum_{s=1}^{S_j} b_{js}(g) < 2$, $g \in \overline{\Gamma_i} \cap \overline{\Gamma_j} \subset \mathcal{K}$, if $i \neq j$ and $\overline{\Gamma_i} \cap \overline{\Gamma_j} \neq \emptyset$.

Now we formulate some auxiliary results to be used in the next sections.

For any closed sets $Q \subset \overline{G}$ and $K \subset \overline{G}$ such that $Q \cap K \neq \emptyset$, we introduce the space

$$C_K(Q) = \{u \in C(Q) : u(y) = 0, \quad y \in Q \cap K\} \quad (2.2)$$

with the maximum-norm. Consider the space of vector-valued functions $\mathcal{C}_\mathcal{K}(\partial G) = \prod_{i=1}^N \mathcal{C}_\mathcal{K}(\overline{\Gamma_i})$ with the norm $\|\psi\|_{\mathcal{C}_\mathcal{K}(\partial G)} = \max_{i=1, \dots, N} \max_{y \in \overline{\Gamma_i}} \|\psi_i\|_{C(\overline{\Gamma_i})}$, where $\psi = \{\psi_i\}$, $\psi_i \in \mathcal{C}_\mathcal{K}(\overline{\Gamma_i})$.

Consider the problem

$$P_0 u - qu = f_0(y), \quad y \in G; \quad u|_{\Gamma_i} - \mathbf{B}_i u = \psi_i(y), \quad y \in \Gamma_i, \quad i = 1, \dots, N. \quad (2.3)$$

Theorem 2.1 (see Theorem 4.1 in [7]). *Let Conditions 2.1–2.3 be fulfilled. Then there is a number $q_1 > 0$ such that, for any $f_0 \in C(\overline{G})$, $\psi = \{\psi_i\} \in \mathcal{C}_K(\partial G)$, and $q \geq q_1$, there exists a unique solution $u \in C_K(\overline{G}) \cap W_{2,\text{loc}}^2(G)$ of problem (2.3). Furthermore, if $f_0 = 0$, then $u \in C_K(\overline{G}) \cap C^\infty(G)$ and the following estimate holds:*

$$\|u\|_{C_K(\overline{G})} \leq c_1 \|\psi\|_{C_K(\partial G)}, \quad (2.4)$$

where $c_1 > 0$ does not depend on ψ and q .

Let $u \in C^\infty(G) \cap C_K(\overline{G})$ be a solution of problem (2.3) with $f_0 = 0$ and $\psi = \{\psi_i\} \in \mathcal{C}_K(\partial G)$. Denote $u = \mathbf{S}_q \psi$. By Theorem 2.1, the operator

$$\mathbf{S}_q : \mathcal{C}_K(\partial G) \rightarrow C_K(\overline{G}), \quad q \geq q_1,$$

is bounded and $\|\mathbf{S}_q\| \leq c_1$, where $c_1 > 0$ does not depend on q .

Lemma 2.1. *Let Conditions 2.1–2.3 hold, let Q_1 and Q_2 be closed sets such that $Q_1 \subset \partial G$, $Q_2 \subset \overline{G}$, and $Q_1 \cap Q_2 = \emptyset$, and let $q \geq q_1$. Then the inequality*

$$\|\mathbf{S}_q \psi\|_{C(Q_2)} \leq \frac{c_2}{q} \|\psi\|_{C_K(\partial G)}, \quad q \geq q_1,$$

holds for any $\psi \in \mathcal{C}_K(\partial G)$ such that $\text{supp}(\mathbf{S}_q \psi)|_{\partial G} \subset Q_1$; here $c_2 > 0$ does not depend on ψ and q .

Proof. Using² Lemma 1.3 in [5] and Theorem 2.1, we obtain

$$\|\mathbf{S}_q \psi\|_{C(Q_2)} \leq \frac{k}{q} \|(\mathbf{S}_q \psi)|_{\partial G}\|_{C(\partial G)} \leq \frac{k}{q} \|\mathbf{S}_q \psi\|_{C(\overline{G})} \leq \frac{k c_1}{q} \|\psi\|_{C_K(\partial G)}, \quad q \geq q_1, \quad (2.5)$$

where the number q_1 defined in Theorem 2.1 is assumed to be large enough so that Lemma 1.3 in [5] be valid for $q \geq q_1$; the number $k = k(q_1)$ does not depend on ψ and q . \square

Lemma 2.2. *Let Conditions 2.1–2.3 hold, let Q_1 and Q_2 be the same sets as in Lemma 2.1, and let $q \geq q_1$. We additionally suppose that $Q_2 \cap \mathcal{K} = \emptyset$. Then the inequality*

$$\|\mathbf{S}_q \psi\|_{C(Q_2)} \leq \frac{c_3}{q} \|\psi\|_{C_K(Q_1)}, \quad q \geq q_1,$$

holds for any $\psi \in \mathcal{C}_K(\partial G)$ such that $\text{supp} \psi \subset Q_1$; here $c_3 > 0$ does not depend on ψ and q .

Proof. 1. Consider a number $\sigma > 0$ such that

$$\text{dist}(Q_1, Q_2) > 3\sigma, \quad \text{dist}(\mathcal{K}, Q_2) > 3\sigma. \quad (2.6)$$

Introduce a function $\xi \in C^\infty(\mathbb{R}^2)$ such that $0 \leq \xi(y) \leq 1$, $\xi(y) = 1$ for $\text{dist}(y, Q_2) \leq \sigma$, and $\xi(y) = 0$ for $\text{dist}(y, Q_2) \geq 2\sigma$.

Consider the auxiliary problem

$$P_0 v - qv = 0, \quad y \in G; \quad v(y) = \xi(y)u(y), \quad y \in \partial G, \quad (2.7)$$

²It is supposed in Lemma 1.3 in [5] that the boundary of domain is infinitely smooth. This assumption is needed to prove the existence of classical solution for elliptic equations with nonhomogeneous boundary condition. However, this assumption is needless for the validity of the first inequality in (2.5), provided that the solution exists.

where $u = \mathbf{S}_q \psi \in C_K(\overline{G})$. Applying Theorem 2.1 with $\mathbf{B}_i = 0$, we see that there is a unique solution $v \in C^\infty(G) \cap C(\overline{G})$ of problem (2.7). It follows from Maximum Principle 2.1 and from the definition of the function ξ that

$$\|v\|_{C(\overline{G})} \leq \|\xi u\|_{C(\partial G)} \leq \max_{i=1,\dots,N} \|u|_{Q_{2,2\sigma} \cap \overline{\Gamma}_i}\|_{C(Q_{2,2\sigma} \cap \overline{\Gamma}_i)}, \quad (2.8)$$

where $Q_{2,2\sigma} = \{y \in \partial G : \text{dist}(y, Q_2) \leq 2\sigma\}$.

Since $\text{supp } \psi \cap Q_{2,2\sigma} = \emptyset$, it follows that

$$u - \mathbf{B}_i u = 0, \quad y \in Q_{2,2\sigma} \cap \overline{\Gamma}_i. \quad (2.9)$$

Taking into account that $\mathbf{B}_i u = 0$ for $y \notin \mathcal{O}_\varepsilon(\mathcal{K})$, we deduce from (2.9) that

$$u(y) = 0, \quad y \in [Q_{2,2\sigma} \cap \overline{\Gamma}_i] \setminus \mathcal{O}_\varepsilon(\mathcal{K}). \quad (2.10)$$

Using (2.8)–(2.10), the definition of the operators \mathbf{B}_i , and Condition 2.3, we obtain

$$\begin{aligned} \|v\|_{C(\overline{G})} &\leq \max_{i=1,\dots,N} \|u|_{Q_{2,2\sigma} \cap \overline{\Gamma}_i \cap \overline{\mathcal{O}_\varepsilon(\mathcal{K})}}\|_{C(Q_{2,2\sigma} \cap \overline{\Gamma}_i \cap \overline{\mathcal{O}_\varepsilon(\mathcal{K})})} \\ &\leq \max_{i=1,\dots,N} \max_{s=1,\dots,S_i} \|u|_{\Omega_{is}(Q_{2,2\sigma} \cap \overline{\Gamma}_i \cap \overline{\mathcal{O}_\varepsilon(\mathcal{K})})}\|_{C(\Omega_{is}(Q_{2,2\sigma} \cap \overline{\Gamma}_i \cap \overline{\mathcal{O}_\varepsilon(\mathcal{K})})}. \end{aligned} \quad (2.11)$$

Since $Q_{2,2\sigma} \cap \mathcal{K} = \emptyset$ (see (2.6)), it follows from the definition of the transformations Ω_{is} that

$$\Omega_{is}(Q_{2,2\sigma} \cap \overline{\Gamma}_i \cap \overline{\mathcal{O}_\varepsilon(\mathcal{K})}) \subset G.$$

Therefore, using inequality (2.11) and Lemma 2.1 with Q_1 and Q_2 replaced by ∂G and $\Omega_{is}(Q_{2,2\sigma} \cap \overline{\Gamma}_i \cap \overline{\mathcal{O}_\varepsilon(\mathcal{K})})$, we have

$$\|v\|_{C(\overline{G})} \leq \frac{c_2}{q} \|\psi\|_{C_K(\partial G)}. \quad (2.12)$$

2. Set $w = u - v$. Clearly, the function w satisfies the relations

$$P_0 w - q w = 0, \quad y \in G; \quad w(y) = u(y) - v(y) = 0, \quad y \in Q_{2,\sigma}.$$

Applying Lemma 2.1 with $\overline{\partial G \setminus Q_{2,\sigma}}$ substituted for Q_1 and $\mathbf{B}_i = 0$ and taking into account that $w|_{\partial G} = (1 - \xi)u|_{\partial G}$, we obtain

$$\|w\|_{C(Q_2)} \leq \frac{c_2}{q} \|w|_{\partial G}\|_{C(\partial G)} \leq \frac{c_2}{q} \|u\|_{C(\overline{G})}.$$

The latter inequality and Theorem 2.1 imply

$$\|w\|_{C(Q_2)} \leq \frac{c_2 c_1}{q} \|\psi\|_{C_K(\partial G)}.$$

Combining this estimate with (2.12), we complete the proof. \square

3 Bounded Perturbations of Elliptic Operators and Their Properties

Introduce a linear operator P_1 satisfying the following condition.

Condition 3.1. The operator $P_1 : C(\overline{G}) \rightarrow C(\overline{G})$ is bounded, and $P_1 u(y^0) \leq 0$ whenever $u \in C(\overline{G})$ achieves its positive maximum at the point $y^0 \in G$.

The operator P_1 will play the role of a bounded perturbation for unbounded elliptic operators in the spaces of continuous functions (cf. [5, 6]).

The following result is a consequence of Conditions 2.1 and 3.1 and Maximum Principle 2.1.

Lemma 3.1. Let Conditions 2.1 and 3.1 hold. If a function $u \in C(\overline{G})$ achieves its positive maximum at a point $y^0 \in G$ and $P_0 u \in C(G)$, then $P_0 u(y^0) + P_1 u(y^0) \leq 0$.

In this paper, we consider the following nonlocal conditions in the *nontransversal* case:

$$b(y)u(y) + \int_{\overline{G}} [u(y) - u(\eta)] \mu(y, d\eta) = 0, \quad y \in \partial G, \quad (3.1)$$

where $b(y) \geq 0$ and $\mu(y, \cdot)$ is a nonnegative Borel measure on \overline{G} .

Set $\mathcal{N} = \{y \in \partial G : \mu(y, \overline{G}) = 0\}$ and $\mathcal{M} = \partial G \setminus \mathcal{N}$. Assume that \mathcal{N} and \mathcal{M} are Borel sets.

Condition 3.2. $\mathcal{K} \subset \mathcal{N}$.

Introduce the function $b_0(y) = b(y) + \mu(y, \overline{G})$.

Condition 3.3. $b_0(y) > 0$ for $y \in \partial G$.

Conditions 3.2 and 3.3 imply that relation (3.1) can be written as follows:

$$u(y) - \int_{\overline{G}} u(\eta) \mu_i(y, d\eta) = 0, \quad y \in \Gamma_i; \quad u(y) = 0, \quad y \in \mathcal{K}, \quad (3.2)$$

where $\mu_i(y, \cdot) = \frac{\mu(y, \cdot)}{b_0(y)}$, $y \in \Gamma_i$. By the definition of the function $b_0(y)$, we have

$$\mu_i(y, \overline{G}) \leq 1, \quad y \in \Gamma_i. \quad (3.3)$$

For any set Q , we denote by $\chi_Q(y)$ the function equal to one on Q and vanishing on $\mathbb{R}^2 \setminus Q$.

Let $b_{is}(y)$ and Ω_{is} be the same as above. We introduce the measures δ_{is} as follows:

$$\delta_{is}(y, Q) = \begin{cases} b_{is}(y) \chi_Q(\Omega_{is}(y)), & y \in \Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K}), \\ 0, & y \in \Gamma_i \setminus \mathcal{O}_\varepsilon(\mathcal{K}), \end{cases}$$

for any Borel set Q .

We study those measures $\mu_i(y, \cdot)$ which can be represented in the form

$$\mu_i(y, \cdot) = \sum_{s=1}^{S_i} \delta_{is}(y, \cdot) + \alpha_i(y, \cdot) + \beta_i(y, \cdot), \quad y \in \Gamma_i, \quad (3.4)$$

where $\alpha_i(y, \cdot)$ and $\beta_i(y, \cdot)$ are nonnegative Borel measures to be specified below (cf. [5, 6]).

For any Borel measure $\mu(y, \cdot)$, the closed set $\text{spt } \mu(y, \cdot) = \overline{G} \setminus \bigcup_{V \in T} \{V \in T : \mu(y, V \cap \overline{G}) = 0\}$ (where T denotes the set of all open sets in \mathbb{R}^2) is called the *support* of the measure $\mu(y, \cdot)$.

Condition 3.4. There exist numbers $\varkappa_1 > \varkappa_2 > 0$ and $\sigma > 0$ such that

1. $\text{spt } \alpha_i(y, \cdot) \subset \overline{G} \setminus \mathcal{O}_{\varkappa_1}(\mathcal{K})$ for $y \in \Gamma_i$,
2. $\text{spt } \alpha_i(y, \cdot) \subset \overline{G}_\sigma$ for $y \in \Gamma_i \setminus \mathcal{O}_{\varkappa_2}(\mathcal{K})$,

where $\mathcal{O}_{\varkappa_1}(\mathcal{K}) = \{y \in \mathbb{R}^2 : \text{dist}(y, \mathcal{K}) < \varkappa_1\}$ and $G_\sigma = \{y \in G : \text{dist}(y, \partial G) < \sigma\}$.

Condition 3.5. $\beta_i(y, \mathcal{M}) < 1$ for $y \in \Gamma_i \cap \mathcal{M}$, $i = 1, \dots, N$.

Remark 3.1. Condition 3.5 is weaker than (analogous) Condition 2.2 in [5] or Condition 3.2 in [6] because the latter two require that $\mu_i(y, \mathcal{M}) < 1$ for $y \in \Gamma_i \cap \mathcal{M}$.

Remark 3.2. One can show that Conditions 3.3–3.5 imply that $b(y) + \mu(y, \overline{G} \setminus \{y\}) > 0$, $y \in \partial G$, i.e., the boundary-value condition (3.1) disappears nowhere on the boundary.

Using relations (3.4), we write nonlocal conditions (3.2) in the form

$$u(y) - \mathbf{B}_i u(y) - \mathbf{B}_{\alpha i} u(y) - \mathbf{B}_{\beta i} u(y) = 0, \quad y \in \Gamma_i; \quad u(y) = 0, \quad y \in \mathcal{K}, \quad (3.5)$$

where the operators \mathbf{B}_i are given by (2.1) and

$$\mathbf{B}_{\alpha i} u(y) = \int_{\overline{G}} u(\eta) \alpha_i(y, d\eta), \quad \mathbf{B}_{\beta i} u(y) = \int_{\overline{G}} u(\eta) \beta_i(y, d\eta), \quad y \in \Gamma_i.$$

Introduce the space³ $C_B(\overline{G}) = \{u \in C(\overline{G}) : u \text{ satisfy nonlocal conditions (3.1)}\}$.

It follows from the definition of the space $C_B(\overline{G})$ and from Condition 3.2 that⁴

$$C_B(\overline{G}) \subset C_{\mathcal{N}}(\overline{G}) \subset C_{\mathcal{K}}(G). \quad (3.6)$$

Lemma 3.2. *Let Conditions 2.1–2.3 and 3.1–3.5 hold. Let a function $u \in C_B(\overline{G})$ achieve its positive maximum at a point $y^0 \in \overline{G}$ and $P_0 u \in C(G)$. Then there is a point $y^1 \in G$ such that $u(y^1) = u(y^0)$ and $P_0 u(y^1) + P_1 u(y^1) \leq 0$.*

Proof. 1. If $y^0 \in G$, then the conclusion of the lemma follows from Lemma 3.1. Let $y^0 \in \partial G$. Suppose that the lemma is not true, i.e., $u(y^0) > u(y)$ for $y \in G$.

Since $u(y^0) > 0$ and $u \in C_B(\overline{G}) \subset C_{\mathcal{N}}(\overline{G})$, it follows that $y^0 \in \mathcal{M}$. Let $y^0 \in \Gamma_i \cap \mathcal{M}$ for some i . If $\mu_i(y^0, G) > 0$, then, taking into account (3.3), we have

$$u(y^0) - \int_{\overline{G}} u(\eta) \mu_i(y^0, d\eta) \geq \int_G [u(y^0) - u(\eta)] \mu_i(y^0, d\eta) > 0,$$

which contradicts (3.2). Therefore, $\text{spt } \mu_i(y^0, \cdot) \subset \partial G$. It follows from this relation, from (3.4), and from Condition 3.4 (part 1) that

$$b_{is}(y^0) = 0, \quad \text{spt } \alpha_i(y^0, \cdot) \subset \partial G \setminus \mathcal{O}_{\varkappa_1}(\mathcal{K}), \quad \text{spt } \beta_i(y^0, \cdot) \subset \partial G. \quad (3.7)$$

2. Suppose that $\alpha_i(y^0, \partial G \setminus \mathcal{O}_{\varkappa_1}(\mathcal{K})) = 0$. In this case, due to (3.7),

$$\alpha_i(y^0, \overline{G}) = 0. \quad (3.8)$$

³Clearly, nonlocal conditions (3.1) in the definition of the space $C_B(\overline{G})$ can be replaced by conditions (3.2) or (3.5).

⁴The spaces $C_{\mathcal{N}}(\cdot)$ and $C_{\mathcal{K}}(\cdot)$ are given in (2.2).

Now it follows from (3.4), (3.7), (3.8) and from Condition 3.5 that

$$\mu_i(y^0, \cdot) = \beta_i(y^0, \cdot), \quad \text{spt } \beta_i(y^0, \cdot) \subset \partial G, \quad \beta_i(y^0, \mathcal{M}) < 1.$$

Hence, the following inequalities hold for $u \in C_B(\overline{G}) \subset C_{\mathcal{N}}(\overline{G})$:

$$u(y^0) - \int_{\overline{G}} u(\eta) \mu_i(y^0, d\eta) = u(y^0) - \int_{\mathcal{M}} u(\eta) \beta_i(y^0, d\eta) \geq u(y^0) - u(y^0) \beta_i(y^0, \mathcal{M}) > 0,$$

which contradicts (3.2).

This contradiction shows that $\alpha_i(y^0, \partial G \setminus \mathcal{O}_{\varkappa_1}(\mathcal{K})) > 0$. Therefore, taking into account Condition 3.4 (part 2), we have $y^0 \in \mathcal{O}_{\varkappa_2}(\mathcal{K})$.

3. We claim that there is a point

$$y' \in \partial G \setminus \mathcal{O}_{\varkappa_1}(\mathcal{K}) \tag{3.9}$$

such that $u(y') = u(y^0)$. Indeed, assume the contrary: $u(y^0) > u(y)$ for $y \in \partial G \setminus \mathcal{O}_{\varkappa_1}(\mathcal{K})$. Then, using (3.3), (3.4), and (3.7), we obtain

$$u(y^0) - \int_{\overline{G}} u(\eta) \mu_i(y^0, d\eta) \geq \int_{\overline{G}} [u(y^0) - u(\eta)] \mu_i(y^0, d\eta) \geq \int_{\partial G \setminus \mathcal{O}_{\varkappa_1}(\mathcal{K})} [u(y^0) - u(\eta)] \alpha_i(y^0, d\eta) > 0 \tag{3.10}$$

because $\alpha_i(y^0, \partial G \setminus \mathcal{O}_{\varkappa_1}(\mathcal{K})) > 0$. Inequality (3.10) contradicts (3.2). Therefore, the function u achieves its positive maximum at some point $y' \in \partial G \setminus \mathcal{O}_{\varkappa_1}(\mathcal{K})$. Repeating the arguments of items 1 and 2 of this proof yields $y' \in \mathcal{O}_{\varkappa_2}(\mathcal{K})$, which contradicts (3.9).

Thus, we have proved that there is a point $y^1 \in G$ such that $u(y^1) = u(y^0)$. Applying Lemma 3.1, we obtain $P_0 u(y^1) + P_1 u(y^1) \leq 0$. \square

Corollary 3.1. *Let Conditions 2.1–2.3 and 3.1–3.5 hold. Let $u \in C_B(\overline{G})$ be a solution of the equation*

$$qu(y) - P_0 u(y) - P_1 u(y) = f_0(y), \quad y \in G,$$

where $q > 0$ and $f_0 \in C(\overline{G})$. Then

$$\|u\|_{C(\overline{G})} \leq \frac{1}{q} \|f_0\|_{C(\overline{G})}. \tag{3.11}$$

Proof. Let $\max_{y \in \overline{G}} |u(y)| = u(y^0) > 0$ for some $y^0 \in \overline{G}$. In this case, by Lemma 3.2, there is a point $y^1 \in G$ such that $u(y^1) = u(y^0)$ and $P_0 u(y^1) + P_1 u(y^1) \leq 0$. Therefore,

$$\|u\|_{C(\overline{G})} = u(y^0) = u(y^1) = \frac{1}{q} (P_0 u(y^1) + P_1 u(y^1) + f_0(y^1)) \leq \frac{1}{q} \|f_0\|_{C(\overline{G})}.$$

\square

4 Reduction to the Operator Equation on the Boundary

In this section, we impose some additional restrictions on the nonlocal operators, which allow us to reduce nonlocal elliptic problems to operator equations on the boundary.

Note that if $u \in C_{\mathcal{N}}(\overline{G})$, then $\mathbf{B}_i u$ is continuous on Γ_i and can be extended to a continuous function on $\overline{\Gamma_i}$ (also denoted by $\mathbf{B}_i u$), which belongs to $C_{\mathcal{N}}(\overline{\Gamma_i})$. We assume that the operators $\mathbf{B}_{\alpha i}$ and $\mathbf{B}_{\beta i}$ possess the similar property.

Condition 4.1. For any function $u \in C_N(\overline{G})$, the functions $\mathbf{B}_{\alpha i}u$ and $\mathbf{B}_{\beta i}u$ can be extended to $\overline{\Gamma_i}$ in such a way that the extended functions (which we also denote by $\mathbf{B}_{\alpha i}u$ and $\mathbf{B}_{\beta i}u$, respectively) belong to $C_N(\overline{\Gamma_i})$.

The next lemma directly follows from the definition of the nonlocal operators.

Lemma 4.1. Let Conditions 2.2, 2.3, 3.2, 3.3, and 4.1 hold. Then the operators $\mathbf{B}_i, \mathbf{B}_{\alpha i}, \mathbf{B}_{\beta i} : C_N(\overline{G}) \rightarrow C_N(\overline{\Gamma_i})$ are bounded and

$$\begin{aligned} \|\mathbf{B}_i u\|_{C_N(\overline{\Gamma_i})} &\leq \|u\|_{C_N(\overline{G})}, & \|\mathbf{B}_{\alpha i} u\|_{C_N(\overline{\Gamma_i})} &\leq \|u\|_{C_N(\overline{G} \setminus \mathcal{O}_{\kappa_1}(\mathcal{K}))}, & \|\mathbf{B}_{\beta i} u\|_{C_N(\overline{\Gamma_i})} &\leq \|u\|_{C_N(\overline{G})}, \\ \|\mathbf{B}_{\alpha i} u + \mathbf{B}_{\beta i} u\| &\leq \|u\|_{C_N(\overline{G})}, & \|\mathbf{B}_i u + \mathbf{B}_{\alpha i} u + \mathbf{B}_{\beta i} u\| &\leq \|u\|_{C_N(\overline{G})}. \end{aligned}$$

Consider the space of vector-valued functions $\mathcal{C}_N(\partial G) = \prod_{i=1}^N C_N(\overline{\Gamma_i})$ with the norm $\|\psi\|_{\mathcal{C}_N(\partial G)} = \max_{i=1, \dots, N} \max_{y \in \overline{\Gamma_i}} \|\psi_i\|_{C(\overline{\Gamma_i})}$, $\psi = \{\psi_i\}$, $\psi_i \in C_N(\overline{\Gamma_i})$.

Introduce the operators

$$\mathbf{B} = \{\mathbf{B}_i\} : C_N(\overline{G}) \rightarrow \mathcal{C}_N(\partial G), \quad \mathbf{B}_{\alpha\beta} = \{\mathbf{B}_{\alpha i} + \mathbf{B}_{\beta i}\} : C_N(\overline{G}) \rightarrow \mathcal{C}_N(\partial G). \quad (4.1)$$

Using the operator \mathbf{S}_q defined in Sec. 2, we introduce the bounded operator

$$\mathbf{I} - \mathbf{B}_{\alpha\beta} \mathbf{S}_q : \mathcal{C}_N(\partial G) \rightarrow \mathcal{C}_N(\partial G), \quad q \geq q_1. \quad (4.2)$$

Since $\mathbf{S}_q \psi \in C_N(\overline{G})$ for $\psi \in \mathcal{C}_N(\partial G)$, the operator in (4.2) is well defined.

Now we formulate sufficient conditions under which the bounded operator $(\mathbf{I} - \mathbf{B}_{\alpha\beta} \mathbf{S}_q)^{-1} : \mathcal{C}_N(\partial G) \rightarrow \mathcal{C}_N(\partial G)$ exists.

We represent the measures $\beta_i(y, \cdot)$ in the form

$$\beta_i(y, \cdot) = \beta_i^1(y, \cdot) + \beta_i^2(y, \cdot), \quad (4.3)$$

where $\beta_i^1(y, \cdot)$ and $\beta_i^2(y, \cdot)$ are nonnegative Borel measures. Let us specify them. For each $p > 0$, we consider the covering of the set $\overline{\mathcal{M}}$ by the p -neighborhoods of all its points. Denote some finite subcovering by \mathcal{M}_p . Since \mathcal{M}_p is a finite union of open disks, it is an open Borel set. Now for each $p > 0$, we consider a cut-off function $\hat{\zeta}_p \in C^\infty(\mathbb{R}^2)$ such that $0 \leq \hat{\zeta}_p(y) \leq 1$, $\hat{\zeta}_p(y) = 1$ for $y \in \mathcal{M}_{p/2}$, and $\hat{\zeta}_p(y) = 0$ for $y \notin \mathcal{M}_p$. Set $\tilde{\zeta}_p = 1 - \hat{\zeta}_p$. Introduce the operators

$$\hat{\mathbf{B}}_{\beta i}^1 u(y) = \int_{\overline{G}} \hat{\zeta}_p(\eta) u(\eta) \beta_i^1(y, d\eta), \quad \tilde{\mathbf{B}}_{\beta i}^1 u(y) = \int_{\overline{G}} \tilde{\zeta}_p(\eta) u(\eta) \beta_i^1(y, d\eta), \quad \mathbf{B}_{\beta i}^2 u(y) = \int_{\overline{G}} u(\eta) \beta_i^2(y, d\eta).$$

Condition 4.2. The following assertions are true for $i = 1, \dots, N$:

1. the operators $\hat{\mathbf{B}}_{\beta i}^1, \tilde{\mathbf{B}}_{\beta i}^1 : C_N(\overline{G}) \rightarrow C_N(\overline{\Gamma_i})$ are bounded;
2. there exists a number $p > 0$ such that⁵

$$\|\hat{\mathbf{B}}_{\beta i}^1\| < \begin{cases} \frac{1}{c_1} & \text{if } \alpha_j(y, \overline{G}) = 0 \ \forall y \in \Gamma_j, \ j = 1, \dots, N, \\ \frac{1}{c_1(1 + c_1)} & \text{otherwise,} \end{cases}$$

where c_1 is the constant occurring in Theorem 2.1.

⁵Part 2 of Condition 4.2 may be replaced by the stronger assumption $\|\hat{\mathbf{B}}_{\beta i}^1\| \rightarrow 0$ as $p \rightarrow 0$, which is easier to verify in applications.

Remark 4.1. The operators $\hat{\mathbf{B}}_{\beta i}^1, \tilde{\mathbf{B}}_{\beta i}^1 : C_{\mathcal{N}}(\overline{G}) \rightarrow C_{\mathcal{N}}(\overline{\Gamma_i})$ are bounded if and only if the operator $\hat{\mathbf{B}}_{\beta i}^1 + \tilde{\mathbf{B}}_{\beta i}^1 : C_{\mathcal{N}}(\overline{G}) \rightarrow C_{\mathcal{N}}(\overline{\Gamma_i})$ is bounded. This follows from the relations $\hat{\mathbf{B}}_{\beta i}^1 u = (\hat{\mathbf{B}}_{\beta i}^1 + \tilde{\mathbf{B}}_{\beta i}^1)(\hat{\zeta}_p u)$ and $\tilde{\mathbf{B}}_{\beta i}^1 u = (\hat{\mathbf{B}}_{\beta i}^1 + \tilde{\mathbf{B}}_{\beta i}^1)(\tilde{\zeta}_p u)$ and from the continuity of the functions $\hat{\zeta}_p$ and $\tilde{\zeta}_p$.

Condition 4.3. The operators $\mathbf{B}_{\beta i}^2 : C_{\mathcal{N}}(\overline{G}) \rightarrow C_{\mathcal{N}}(\overline{\Gamma_i})$, $i = 1, \dots, N$, are compact.

It follows from (3.4) and (4.3) that the measures $\mu_i(y, \cdot)$ have the following representation:

$$\mu_i(y, \cdot) = \sum_{s=1}^{S_i} \delta_{is}(y, \cdot) + \alpha_i(y, \cdot) + \beta_i^1(y, \cdot) + \beta_i^2(y, \cdot), \quad y \in \Gamma_i.$$

The measures $\delta_{is}(y, \cdot)$ correspond to nonlocal terms supported near the set \mathcal{K} of the conjugation points. The measures $\alpha_i(y, \cdot)$ correspond to nonlocal terms supported outside the set \mathcal{K} . The measures $\beta_i^1(y, \cdot)$ and $\beta_i^2(y, \cdot)$ correspond to nonlocal terms with arbitrary geometrical structure of their support (in particular, their support may intersect with the set \mathcal{K}); however, the measure $\beta_i^1(y, \mathcal{M}_p)$ of the set \mathcal{M}_p must be small for small p (Condition 4.2) and the measure $\beta_i^2(y, \cdot)$ must generate a compact operator (Condition 4.3).

Lemma 4.2. Let Conditions 2.1–2.3, 3.1–3.5, and 4.1–4.3 hold. Then there exists a bounded operator $(\mathbf{I} - \mathbf{B}_{\alpha\beta}\mathbf{S}_q)^{-1} : \mathcal{C}_{\mathcal{N}}(\partial G) \rightarrow \mathcal{C}_{\mathcal{N}}(\partial G)$, $q \geq q_1$, where $q_1 > 0$ is sufficiently large.

Proof. 1. Consider the bounded operators $\hat{\mathbf{B}}_{\beta}^1 = \{\hat{\mathbf{B}}_{\beta i}^1\}$, $\tilde{\mathbf{B}}_{\beta}^1 = \{\tilde{\mathbf{B}}_{\beta i}^1\}$, $\mathbf{B}_{\beta}^2 = \{\mathbf{B}_{\beta i}^2\}$, and $\mathbf{B}_{\alpha} = \{\mathbf{B}_{\alpha i}\}$ acting from $C_{\mathcal{N}}(\overline{G})$ to $\mathcal{C}_{\mathcal{N}}(\partial G)$ (cf. (4.1)).

Let us prove that the operator $\mathbf{I} - \mathbf{B}_{\alpha}\mathbf{S}_q : \mathcal{C}_{\mathcal{N}}(\partial G) \rightarrow \mathcal{C}_{\mathcal{N}}(\partial G)$ has the bounded inverse.

Introduce a function $\zeta \in C^{\infty}(\overline{G})$ such that $0 \leq \zeta(y) \leq 1$, $\zeta(y) = 1$ for $y \in \overline{G_{\sigma}}$, and $\zeta(y) = 0$ for $y \notin G_{\sigma/2}$, where $\sigma > 0$ is the number from Condition 3.4.

We have

$$\mathbf{I} - \mathbf{B}_{\alpha}\mathbf{S}_q = \mathbf{I} - \mathbf{B}_{\alpha}(1 - \zeta)\mathbf{S}_q - \mathbf{B}_{\alpha}\zeta\mathbf{S}_q. \quad (4.4)$$

1a. First, we show that the operator $\mathbf{I} - \mathbf{B}_{\alpha}(1 - \zeta)\mathbf{S}_q$ has the bounded inverse. By Lemma 4.1 and Theorem 2.1,

$$\|\mathbf{B}_{\alpha}(1 - \zeta)\mathbf{S}_q\| \leq c_1. \quad (4.5)$$

Furthermore, $(1 - \zeta)\mathbf{S}_q\psi = 0$ in $\overline{G_{\sigma}}$ for any $\psi \in \mathcal{C}_{\mathcal{N}}(\partial G)$. Therefore, by Condition 3.4,

$$\text{supp } \mathbf{B}_{\alpha}(1 - \zeta)\mathbf{S}_q\psi \subset \partial G \cap \overline{\mathcal{O}_{\kappa_2}(\mathcal{K})}. \quad (4.6)$$

Let us show that

$$\|[\mathbf{B}_{\alpha}(1 - \zeta)\mathbf{S}_q]^2\| \leq \frac{c}{q}, \quad q \geq q_1, \quad (4.7)$$

where $q_1 > 0$ is sufficiently large and $c > 0$ does not depend on q . Consecutively applying (I) Lemma 4.1, (II) Lemma 2.2 and relation (4.6), and (III) Lemma 4.1 and Theorem 2.1, we obtain

$$\begin{aligned} \|\mathbf{B}_{\alpha}(1 - \zeta)\mathbf{S}_q \mathbf{B}_{\alpha}(1 - \zeta)\mathbf{S}_q\psi\|_{\mathcal{C}_{\mathcal{N}}(\partial G)} &\leq \|\mathbf{S}_q \mathbf{B}_{\alpha}(1 - \zeta)\mathbf{S}_q\psi\|_{C_{\mathcal{N}}(\overline{G} \setminus \mathcal{O}_{\kappa_1}(\mathcal{K}))} \leq \\ &\leq \frac{c_3 c_1}{q} \|\mathbf{B}_{\alpha}(1 - \zeta)\mathbf{S}_q\psi\|_{C_{\mathcal{N}}(\partial G \cap \overline{\mathcal{O}_{\kappa_2}(\mathcal{K})})} \leq \frac{c_3 c_1}{q} \|\psi\|_{\mathcal{C}_{\mathcal{N}}(\partial G)}. \end{aligned}$$

This yields (4.7) with $c = c_3 c_1$.

If $q \geq 2c$, then the operator $\mathbf{I} - [\mathbf{B}_{\alpha}(1 - \zeta)\mathbf{S}_q]^2$ has the bounded inverse. Therefore, the operator $\mathbf{I} - \mathbf{B}_{\alpha}(1 - \zeta)\mathbf{S}_q$ also has the bounded inverse and

$$[\mathbf{I} - \mathbf{B}_{\alpha}(1 - \zeta)\mathbf{S}_q]^{-1} = [\mathbf{I} + \mathbf{B}_{\alpha}(1 - \zeta)\mathbf{S}_q][\mathbf{I} - (\mathbf{B}_{\alpha}(1 - \zeta)\mathbf{S}_q)^2]^{-1}. \quad (4.8)$$

Representation (4.8), Lemma 4.1, Theorem 2.1 and relations (4.5) and (4.7) imply that

$$\|[\mathbf{I} - \mathbf{B}_\alpha(1 - \zeta)\mathbf{S}_q]^{-1}\| = 1 + c_1 + O(q^{-1}), \quad q \rightarrow +\infty. \quad (4.9)$$

1b. Now we estimate the norm of the operator $\mathbf{B}_\alpha\zeta\mathbf{S}_q$. Lemmas 4.1 and 2.2 imply that

$$\|\mathbf{B}_\alpha\zeta\mathbf{S}_q\psi\|_{\mathcal{C}_N(\partial G)} \leq \|\mathbf{S}_q\psi\|_{C(\overline{G_{\sigma/2}})} \leq \frac{c_2}{q}\|\psi\|_{\mathcal{C}_N(\partial G)}. \quad (4.10)$$

Therefore, using representation (4.4), we see that the operator $\mathbf{I} - \mathbf{B}_\alpha\mathbf{S}_q$ has the bounded inverse for sufficiently large q and

$$(\mathbf{I} - \mathbf{B}_\alpha\mathbf{S}_q)^{-1} = [\mathbf{I} - (\mathbf{I} - \mathbf{B}_\alpha(1 - \zeta)\mathbf{S}_q)^{-1}\mathbf{B}_\alpha\zeta\mathbf{S}_q]^{-1}[\mathbf{I} - \mathbf{B}_\alpha(1 - \zeta)\mathbf{S}_q]^{-1}. \quad (4.11)$$

It follows from (4.9)–(4.11) that

$$\|(\mathbf{I} - \mathbf{B}_\alpha\mathbf{S}_q)^{-1}\| = 1 + c_1 + O(q^{-1}), \quad q \rightarrow +\infty. \quad (4.12)$$

2. Let us prove that the operator $\mathbf{I} - (\mathbf{B}_\alpha + \hat{\mathbf{B}}_\beta^1 + \tilde{\mathbf{B}}_\beta^1)\mathbf{S}_q : \mathcal{C}_N(\partial G) \rightarrow \mathcal{C}_N(\partial G)$ has the bounded inverse.

2a. It follows from the definition of the operator $\tilde{\mathbf{B}}_\beta^1$ and from Lemma 2.1 (with $Q_1 = \overline{\mathcal{M}}$ and $Q_2 = \overline{G} \setminus \mathcal{M}_{p/2}$) that

$$\|\tilde{\mathbf{B}}_{\beta i}^1\mathbf{S}_q\psi\|_{\mathcal{C}_N(\overline{\Gamma_i})} \leq \|\mathbf{S}_q\psi\|_{C(\overline{G} \setminus \mathcal{M}_{p/2})} \leq \frac{c_2}{q}\|\psi\|_{\mathcal{C}_N(\partial G)} \quad (4.13)$$

because $(\overline{G} \setminus \mathcal{M}_{p/2}) \cap \overline{\mathcal{M}} = \emptyset$ and $\text{supp}(\mathbf{S}_q\psi)|_{\partial G} \subset \overline{\mathcal{M}}$ for $\psi \in \mathcal{C}_N(\partial G)$.

2b. Let $\alpha_j(y, \overline{G}) \neq 0$ for some j and $y \in \Gamma_j$. Due to Condition 4.2 (part 2) and Theorem 2.1, there is a number d such that $0 < 2d < 1/(1 + c_1)$ and

$$\|\hat{\mathbf{B}}_{\beta i}^1\mathbf{S}_q\psi\|_{\mathcal{C}_N(\overline{\Gamma_i})} \leq \left(\frac{1}{c_1(1 + c_1)} - \frac{2d}{c_1}\right)\|\mathbf{S}_q\psi\|_{\mathcal{C}_N(\overline{G})} \leq \left(\frac{1}{1 + c_1} - 2d\right)\|\psi\|_{\mathcal{C}_N(\partial G)}. \quad (4.14)$$

Inequalities (4.13) and (4.14) yield

$$\|(\hat{\mathbf{B}}_\beta^1 + \tilde{\mathbf{B}}_\beta^1)\mathbf{S}_q\| \leq \frac{1}{1 + c_1} - d \quad (4.15)$$

for sufficiently large q . Now it follows from (4.12) and (4.15) that $\|(\mathbf{I} - \mathbf{B}_\alpha\mathbf{S}_q)^{-1}(\hat{\mathbf{B}}_\beta^1 + \tilde{\mathbf{B}}_\beta^1)\mathbf{S}_q\| < 1$ for sufficiently large q . Hence, there exists the bounded inverse operator

$$[\mathbf{I} - (\mathbf{B}_\alpha + \hat{\mathbf{B}}_\beta^1 + \tilde{\mathbf{B}}_\beta^1)\mathbf{S}_q]^{-1} = [\mathbf{I} - (\mathbf{I} - \mathbf{B}_\alpha\mathbf{S}_q)^{-1}(\hat{\mathbf{B}}_\beta^1 + \tilde{\mathbf{B}}_\beta^1)\mathbf{S}_q]^{-1}[\mathbf{I} - \mathbf{B}_\alpha\mathbf{S}_q]^{-1}. \quad (4.16)$$

2c. If $\alpha_j(y, \overline{G}) = 0$ for $y \in \Gamma_j$, $j = 1, \dots, N$, then, due to Condition 4.2 (part 1), inequality (4.14) assumes the form

$$\|\hat{\mathbf{B}}_{\beta i}^1\mathbf{S}_q\psi\|_{\mathcal{C}_N(\overline{\Gamma_i})} \leq \left(\frac{1}{c_1} - \frac{2d}{c_1}\right)\|\mathbf{S}_q\psi\|_{\mathcal{C}_N(\overline{G})} \leq (1 - 2d)\|\psi\|_{\mathcal{C}_N(\partial G)}.$$

Therefore, inequality (4.15) reduces to

$$\|(\hat{\mathbf{B}}_\beta^1 + \tilde{\mathbf{B}}_\beta^1)\mathbf{S}_q\| \leq 1 - d. \quad (4.17)$$

Since $\mathbf{B}_\alpha = 0$ in the case under consideration, it follows from (4.17) that the operator

$$\mathbf{I} - (\mathbf{B}_\alpha + \hat{\mathbf{B}}_\beta^1 + \tilde{\mathbf{B}}_\beta^1)\mathbf{S}_q = \mathbf{I} - (\hat{\mathbf{B}}_\beta^1 + \tilde{\mathbf{B}}_\beta^1)\mathbf{S}_q$$

has the bounded inverse.

3. It remains to show that the operator $\mathbf{I} - \mathbf{B}_{\alpha\beta}\mathbf{S}_q$ also has the bounded inverse. By Condition 4.3, the operator \mathbf{B}_β^2 is compact. Therefore, the operator $\mathbf{B}_\beta^2\mathbf{S}_q$ is also compact. Since the index of a Fredholm operator is stable under compact perturbation, we see that the operator $\mathbf{I} - \mathbf{B}_{\alpha\beta}\mathbf{S}_q$ has the Fredholm property and $\text{ind}(\mathbf{I} - \mathbf{B}_{\alpha\beta}\mathbf{S}_q) = 0$. To prove that $\mathbf{I} - \mathbf{B}_{\alpha\beta}\mathbf{S}_q$ has the bounded inverse, it now suffices to show that $\dim \ker(\mathbf{I} - \mathbf{B}_{\alpha\beta}\mathbf{S}_q) = 0$.

Let $\psi \in \mathcal{C}_\mathcal{N}(\partial G)$ and $(\mathbf{I} - \mathbf{B}_{\alpha\beta}\mathbf{S}_q)\psi = 0$. Then the function $u = \mathbf{S}_q\psi \in C^\infty(G) \cap C_\mathcal{N}(\overline{G})$ is a solution of the problem

$$\begin{aligned} P_0u - qu &= 0, \quad y \in G, \\ u(y) - \mathbf{B}_i u(y) - \mathbf{B}_{\alpha i} u(y) - \mathbf{B}_{\beta i} u(y) &= 0, \quad y \in \Gamma_i; \quad u(y) = 0, \quad y \in \mathcal{K}. \end{aligned}$$

By Corollary 3.1, we have $u = 0$. Therefore, $\psi = \mathbf{B}_{\alpha\beta}\mathbf{S}_q\psi = \mathbf{B}_{\alpha\beta}u = 0$. \square

5 Existence of Feller Semigroups

In this section, we prove that the above bounded perturbations of elliptic equations with nonlocal conditions satisfying hypotheses of Secs. 2–4 are generators of some Feller semigroups.

Reducing nonlocal problems to the boundary and using Lemma 4.2, we prove that the nonlocal problems are solvable in the space of continuous functions.

Lemma 5.1. *Let Conditions 2.1–2.3, 3.2–3.5, and 4.1–4.3 hold, and let q_1 be sufficiently large. Then, for any $q \geq q_1$ and $f_0 \in C(\overline{G})$, the problem*

$$qu(y) - P_0u(y) = f_0(y), \quad y \in G, \quad (5.1)$$

$$u(y) - \mathbf{B}_i u(y) - \mathbf{B}_{\alpha i} u(y) - \mathbf{B}_{\beta i} u(y) = 0, \quad y \in \Gamma_i; \quad u(y) = 0, \quad y \in \mathcal{K}, \quad (5.2)$$

admits a unique solution $u \in C_B(\overline{G}) \cap W_{2,\text{loc}}^2(G)$.

Proof. Let us consider the auxiliary problem

$$qv(y) - P_0v(y) = f_0(y), \quad y \in G; \quad v(y) - \mathbf{B}_i v(y) = 0, \quad y \in \Gamma_i, \quad i = 1, \dots, N. \quad (5.3)$$

Since $f_0 \in C(\overline{G})$, it follows from Theorem 2.1 that there exists a unique solution $v \in C_\mathcal{K}(\overline{G})$ of problem (5.3). Therefore, $v \in C_\mathcal{N}(\overline{G})$.

2. Set $w = u - v$. The unknown function w belongs to $C_\mathcal{N}(\overline{G})$, and, by virtue of (5.1)–(5.3), it satisfies the relations

$$\begin{aligned} qw(y) - P_0w(y) &= 0, & y \in G, \\ w(y) - \mathbf{B}_i w(y) - \mathbf{B}_{\alpha i} w(y) - \mathbf{B}_{\beta i} w(y) &= \mathbf{B}_{\alpha i} v(y) + \mathbf{B}_{\beta i} v(y), & y \in \Gamma_i, \quad i = 1, \dots, N, \\ w(y) &= 0, & y \in \mathcal{K}. \end{aligned} \quad (5.4)$$

It follows from Condition 4.1 that problem (5.4) is equivalent to the operator equation $\psi - \mathbf{B}_{\alpha\beta}\mathbf{S}_q\psi = \mathbf{B}_{\alpha\beta}v$ for the unknown function $\psi \in \mathcal{C}_\mathcal{N}(\partial G)$. Lemma 4.2 implies that this equation admits a unique solution $\psi \in \mathcal{C}_\mathcal{N}(\partial G)$. In this case, problem (5.1), (5.2) admits a unique solution

$$u = v + w = v + \mathbf{S}_q\psi = v + \mathbf{S}_q(\mathbf{I} - \mathbf{B}_{\alpha\beta}\mathbf{S}_q)^{-1}\mathbf{B}_{\alpha\beta}v \in C_B(\overline{G}).$$

Moreover, $u \in W_{2,\text{loc}}^2(G)$ due to the interior regularity theorem for elliptic equations. \square

Using Lemma 5.1 and the assumptions concerning the bounded perturbations (see Condition 3.1), we prove that the perturbed problems are solvable in the space of continuous functions.

Lemma 5.2. *Let Conditions 2.1–2.3, 3.1–3.5, and 4.1–4.3 hold, and let q_1 be sufficiently large. Then, for any $q \geq q_1$ and $f_0 \in C(\overline{G})$, the problem*

$$qu - (P_0 + P_1)u = f_0(y), \quad y \in G, \quad (5.5)$$

$$u(y) - \mathbf{B}_i u(y) - \mathbf{B}_{\alpha i} u(y) - \mathbf{B}_{\beta i} u(y) = 0, \quad y \in \Gamma_i; \quad u(y) = 0, \quad y \in \mathcal{K}, \quad (5.6)$$

admits a unique solution $u \in C_B(\overline{G}) \cap W_{2,\text{loc}}^2(G)$.

Proof. Consider the operator $qI - P_0$ as the operator acting from $C(\overline{G})$ to $C(\overline{G})$ with the domain

$$D(qI - P_0) = \{u \in C_B(\overline{G}) \cap W_{2,\text{loc}}^2(G) : P_0 u \in C(\overline{G})\}.$$

Lemma 5.1 and Corollary 3.1 imply that there exists the bounded operator $(qI - P_0)^{-1} : C(\overline{G}) \rightarrow C(\overline{G})$ and

$$\|(qI - P_0)^{-1}\| \leq 1/q.$$

Introduce the operator $qI - P_0 - P_1 : C(\overline{G}) \rightarrow C(\overline{G})$ with the domain $D(qI - P_0 - P_1) = D(qI - P_0)$. Since

$$qI - P_0 - P_1 = (I - P_1(qI - P_0)^{-1})(qI - P_0),$$

it follows that the operator $qI - P_0 - P_1 : C(\overline{G}) \rightarrow C(\overline{G})$ has the bounded inverse for $q \geq q_1$, provided that q_1 is so large that $\|P_1\| \cdot \|(qI - P_0)^{-1}\| \leq 1/2$, $q \geq q_1$. \square

We consider the unbounded operator $\mathbf{P}_B : D(\mathbf{P}_B) \subset C_B(\overline{G}) \rightarrow C_B(\overline{G})$ given by

$$\mathbf{P}_B u = P_0 u + P_1 u, \quad u \in D(\mathbf{P}_B) = \{u \in C_B(\overline{G}) \cap W_{2,\text{loc}}^2(G) : P_0 u + P_1 u \in C_B(\overline{G})\}. \quad (5.7)$$

Lemma 5.3. *Let Conditions 2.1–2.3, 3.1–3.5, and 4.1–4.3 hold. Then the set $D(\mathbf{P}_B)$ is dense in $C_B(\overline{G})$.*

Proof. We will follow the scheme proposed in [6].

1. Let $u \in C_B(\overline{G})$. Since $C_B(\overline{G}) \subset C_{\mathcal{N}}(\overline{G})$ due to (3.6), it follows that, for any $\varepsilon > 0$ and $q \geq q_1$, there is a function $u_1 \in C^\infty(\overline{G}) \cap C_{\mathcal{N}}(\overline{G})$ such that

$$\|u - u_1\|_{C(\overline{G})} \leq \min(\varepsilon, \varepsilon/(2c_1 k_q)), \quad (5.8)$$

where $k_q = \|(\mathbf{I} - \mathbf{B}_{\alpha\beta} \mathbf{S}_q)^{-1}\|$.

Set

$$\begin{aligned} f_0(y) &\equiv qu_1 - P_0 u_1, & y \in G, \\ \psi_i(y) &\equiv u_1(y) - \mathbf{B}_i u_1(y) - \mathbf{B}_{\alpha i} u_1(y) - \mathbf{B}_{\beta i} u_1(y), & y \in \Gamma_i, \quad i = 1, \dots, N. \end{aligned} \quad (5.9)$$

Since $u_1 \in C_{\mathcal{N}}(\overline{G})$, it follows from Condition 4.1 that $\{\psi_i\} \in \mathcal{C}_{\mathcal{N}}(\partial G)$. Using the relation

$$u(y) - \mathbf{B}_i u(y) - \mathbf{B}_{\alpha i} u(y) - \mathbf{B}_{\beta i} u(y) = 0, \quad y \in \Gamma_i,$$

inequality (5.8), and Lemma 4.1, we obtain

$$\|\{\psi_i\}\|_{\mathcal{C}_{\mathcal{N}}(\partial G)} \leq \|u - u_1\|_{C(\overline{G})} + \|(\mathbf{B} + \mathbf{B}_{\alpha\beta})(u - u_1)\|_{\mathcal{C}_{\mathcal{N}}(\partial G)} \leq \varepsilon/(c_1 k_q). \quad (5.10)$$

Consider the auxiliary nonlocal problem

$$\begin{aligned} qu_2 - P_0 u_2 &= f_0(y), \quad y \in G, \\ u_2(y) - \mathbf{B}_i u_2(y) - \mathbf{B}_{\alpha i} u_2(y) - \mathbf{B}_{\beta i} u_2(y) &= 0, \quad y \in \Gamma_i; \quad u_2(y) = 0, \quad y \in \mathcal{K}. \end{aligned} \quad (5.11)$$

Since $f_0 \in C^\infty(\overline{G})$, it follows from Lemma 5.1 that problem (5.11) has a unique solution $u_2 \in C_B(\overline{G}) \subset C_{\mathcal{N}}(\overline{G})$.

Using (5.9), (5.11), and the relations $u_1(y) = u_2(y) = 0$, $y \in \mathcal{K}$, we see that the function $w_1 = u_1 - u_2$ satisfies the relations

$$\begin{aligned} qw_1 - P_0 w_1 &= 0, \quad y \in G, \\ w_1(y) - \mathbf{B}_i w_1(y) - \mathbf{B}_{\alpha i} w_1(y) - \mathbf{B}_{\beta i} w_1(y) &= \psi_i(y), \quad y \in \Gamma_i; \quad w_1(y) = 0, \quad y \in \mathcal{K}. \end{aligned} \quad (5.12)$$

It follows from Condition 4.1 that problem (5.12) is equivalent to the operator equation $\varphi - \mathbf{B}_{\alpha\beta} \mathbf{S}_q \varphi = \psi$ in $\mathcal{C}_{\mathcal{N}}(\partial G)$, where $w_1 = \mathbf{S}_q \varphi$. Lemma 4.2 implies that this equation admits a unique solution $\varphi \in \mathcal{C}_{\mathcal{N}}(\partial G)$. Therefore, using Theorem 2.1 and inequality (5.10), we obtain

$$\|w_1\|_{C(\overline{G})} \leq c_1 \|(\mathbf{I} - \mathbf{B}_{\alpha\beta} \mathbf{S}_q)^{-1}\| \cdot \|\{\psi_i\}\|_{\mathcal{C}_{\mathcal{N}}(\partial G)} \leq c_1 k_q \varepsilon / (c_1 k_q) = \varepsilon. \quad (5.13)$$

2. Finally, we consider the problem

$$\begin{aligned} \lambda u_3 - P_0 u_3 - P_1 u_3 &= \lambda u_2, \quad y \in G, \\ u_3(y) - \mathbf{B}_i u_3(y) - \mathbf{B}_{\alpha i} u_3(y) - \mathbf{B}_{\beta i} u_3(y) &= 0, \quad y \in \Gamma_i; \quad u_3(y) = 0, \quad y \in \mathcal{K}. \end{aligned} \quad (5.14)$$

Since $u_2 \in C_B(\overline{G})$, it follows from Lemma 5.2 that problem (5.14) admits a unique solution $u_3 \in D(\mathbf{P}_B)$ for sufficiently large λ .

Denote $w_2 = u_2 - u_3$. It follows from (5.14) that

$$\lambda w_2 - P_0 w_2 - P_1 w_2 = -P_0 u_2 - P_1 u_2 = f_0 - qu_2 - P_1 u_2.$$

Applying Corollary 3.1, we have

$$\|w_2\|_{C(\overline{G})} \leq \frac{1}{\lambda} \|f_0 - qu_2 - P_1 u_2\|_{C(\overline{G})}.$$

Choosing sufficiently large λ yields

$$\|w_2\|_{C(\overline{G})} \leq \varepsilon. \quad (5.15)$$

Inequalities (5.8), (5.13), and (5.15) imply

$$\|u - u_3\|_{C(\overline{G})} \leq \|u - u_1\|_{C(\overline{G})} + \|u_1 - u_2\|_{C(\overline{G})} + \|u_2 - u_3\|_{C(\overline{G})} \leq 3\varepsilon.$$

□

Now we can prove the main result of the paper.

Theorem 5.1. *Let Conditions 2.1–2.3, 3.1–3.5, and 4.1–4.3 hold. Then the operator $\mathbf{P}_B : D(\mathbf{P}_B) \subset C_B(\overline{G}) \rightarrow C_B(\overline{G})$ is a generator of a Feller semigroup.*

Proof. 1. By Lemma 5.2 and Corollary 3.1, there exists the bounded operator $(qI - \mathbf{P}_B)^{-1} : C_B(\overline{G}) \rightarrow C_B(\overline{G})$ and

$$\|(qI - \mathbf{P}_B)^{-1}\| \leq 1/q$$

for all sufficiently large $q > 0$.

2. Since the operator $(qI - \mathbf{P}_B)^{-1}$ is bounded and defined on the whole space $C_B(\overline{G})$, it is closed. Therefore, the operator $qI - \mathbf{P}_B : D(\mathbf{P}_B) \subset C_B(\overline{G}) \rightarrow C_B(\overline{G})$ is closed. Hence, $\mathbf{P}_B : D(\mathbf{P}_B) \subset C_B(\overline{G}) \rightarrow C_B(\overline{G})$ is also closed.

3. Let us prove that the operator $(qI - \mathbf{P}_B)^{-1}$ is nonnegative. Assume the contrary; then there exists a function $f_0 \geq 0$ such that a solution $u \in D(\mathbf{P}_B)$ of the equation $qu - \mathbf{P}_B u = f_0$ achieves its negative minimum at some point $y^0 \in \overline{G}$. In this case, the function $v = -u$ achieves its positive maximum at the point y^0 . By Lemma 3.2, there is a point $y^1 \in G$ such that $v(y^1) = v(y^0)$ and $\mathbf{P}_B v(y^1) \leq 0$. Therefore, $0 < v(y^0) = v(y^1) = (\mathbf{P}_B v(y^1) - f_0(y^1))/q \leq 0$. This contradiction proves that $u \geq 0$.

Thus, all the hypotheses of the Hille–Iosida theorem (Theorem 1.1) are fulfilled. Hence, $\mathbf{P}_B : D(\mathbf{P}_B) \subset C_B(\overline{G}) \rightarrow C_B(\overline{G})$ is a generator of a Feller semigroup. \square

As a conclusion, we give an example of nonlocal conditions satisfying the assumptions of the paper.

Let $G \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial G = \Gamma_1 \cup \Gamma_2 \cup \mathcal{K}$, where Γ_1 and Γ_2 are C^∞ curves open and connected in the topology of ∂G such that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \mathcal{K}$; the set \mathcal{K} consists of two points g_1 and g_2 . We assume that the domain G coincides with some plane angle in an ε -neighborhood of the point g_i , $i = 1, 2$. Let Ω_j , $j = 1, \dots, 4$, be continuous transformations defined on $\overline{\Gamma_1}$ and satisfying the following conditions (see Fig. 5.1):

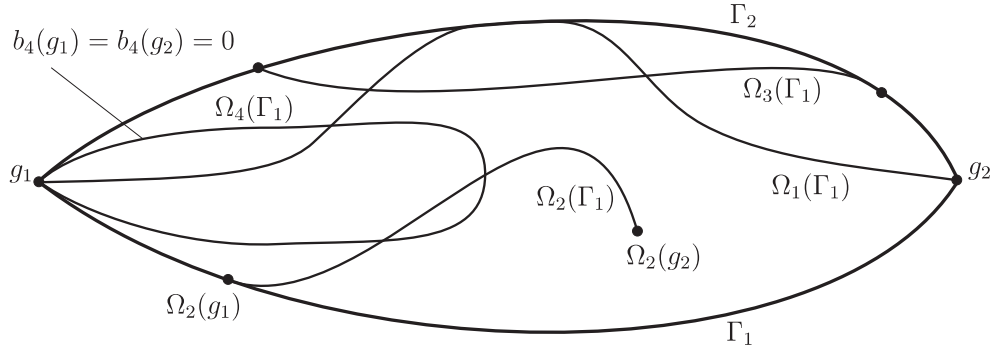


Figure 5.1: Nontransversal nonlocal conditions

1. $\Omega_1(\mathcal{K}) \subset \mathcal{K}$, $\Omega_1(\Gamma_1 \cap \mathcal{O}_\varepsilon(\mathcal{K})) \subset G$, $\Omega_1(\Gamma_1 \setminus \mathcal{O}_\varepsilon(\mathcal{K})) \subset G \cup \Gamma_2$, and $\Omega_1(y)$ is a composition of shift of the argument, rotation, and homothety for $y \in \overline{\Gamma_1} \cap \mathcal{O}_\varepsilon(\mathcal{K})$;
2. there exist numbers $\varkappa_1 > \varkappa_2 > 0$ and $\sigma > 0$ such that $\Omega_2(\overline{\Gamma_1}) \subset \overline{G} \setminus \mathcal{O}_{\varkappa_1}(\mathcal{K})$ and $\Omega_2(\overline{\Gamma_1} \setminus \mathcal{O}_{\varkappa_2}(\mathcal{K})) \subset \overline{G}_\sigma$; moreover, $\Omega_2(g_1) \in \Gamma_1$ and $\Omega_2(g_2) \in G$;
3. $\Omega_3(\overline{\Gamma_1}) \subset G \cup \Gamma_2$ and $\Omega_3(\mathcal{K}) \subset \Gamma_2$;
4. $\Omega_4(\overline{\Gamma_1}) \subset G \cup \overline{\Gamma_2}$ and $\Omega_4(\mathcal{K}) \subset \mathcal{K}$.

Let $b_1 \in C(\overline{\Gamma_1}) \cap C^\infty(\overline{\Gamma_1} \cap \mathcal{O}_\varepsilon(\mathcal{K}))$, $b_2, b_3, b_4 \in C(\overline{\Gamma_1})$, and $b_j \geq 0$, $j = 1, \dots, 4$.

Let G_1 be a bounded domain, $G_1 \subset G$, and $\Gamma \subset \overline{G}$ be a curve of class C^1 . Introduce continuous nonnegative functions $c(y, \eta)$, $y \in \overline{\Gamma_1}$, $\eta \in \overline{G_1}$, and $d(y, \eta)$, $y \in \overline{\Gamma_1}$, $\eta \in \overline{\Gamma}$.

Consider the following nonlocal conditions:

$$\begin{aligned} u(y) - \sum_{j=1}^4 b_j(y)u(\Omega_j(y)) - \int_{G_1} c(y, \eta)u(\eta)d\eta - \int_{\Gamma} d(y, \eta)u(\eta)d\Gamma_{\eta} &= 0, \quad y \in \Gamma_1, \\ u(y) &= 0, \quad y \in \overline{\Gamma_2}. \end{aligned} \tag{5.16}$$

Let $Q \subset \overline{G}$ be an arbitrary Borel set; introduce the measure $\mu(y, \cdot)$, $y \in \partial G$:

$$\begin{aligned} \mu(y, Q) &= \sum_{j=1}^4 b_j(y)\chi_Q(\Omega_j(y)) + \int_{G_1 \cap Q} c(y, \eta)d\eta + \int_{\Gamma \cap Q} d(y, \eta)u(\eta)d\Gamma_{\eta}, \quad y \in \Gamma_1, \\ \mu(y, Q) &= 0, \quad y \in \overline{\Gamma_2}, \end{aligned}$$

Let \mathcal{N} and \mathcal{M} be defined as before. Assume that

$$\begin{aligned} \mu(y, \overline{G}) &= \sum_{j=1}^4 b_j(y) + \int_{G_1} c(y, \eta)d\eta + \int_{\Gamma} d(y, \eta)d\Gamma_{\eta} \leq 1, \quad y \in \partial G, \\ \int_{\Gamma \cap \mathcal{M}} d(y, \eta)d\Gamma_{\eta} &< 1, \quad y \in \mathcal{M}; \\ b_2(g_1) &= 0 \text{ or } \mu(\Omega_2(g_1), \overline{G}) = 0, \quad b_2(g_2) = 0; \quad b_4(g_j) = 0; \quad c(g_j, \cdot) = 0; \quad d(g_j, \cdot) = 0. \end{aligned}$$

Setting $b(y) = 1 - \mu(y, \overline{G})$, we can rewrite (5.16) in the form (cf. (3.1))

$$b(y)u(y) + \int_{\overline{G}} [u(y) - u(\eta)]\mu(y, d\eta) = 0, \quad y \in \partial G.$$

Introduce a cut-off function $\zeta \in C^\infty(\mathbb{R}^2)$ supported in $\mathcal{O}_\varepsilon(\mathcal{K})$, equal to 1 on $\mathcal{O}_{\varepsilon/2}(\mathcal{K})$, and such that $0 \leq \zeta(y) \leq 1$ for $y \in \mathbb{R}^2$. Let $y \in \overline{\Gamma_1}$ and $Q \subset \overline{G}$ be a Borel set; denote

$$\begin{aligned} \delta(y, Q) &= \zeta(y)b_1(y)\chi_Q(\Omega_1(y)), \quad \alpha(y, Q) = b_2(y)\chi_Q(\Omega_2(y)), \\ \beta^1(y, Q) &= (1 - \zeta(y))b_1(y)\chi_Q(\Omega_1(y)) + \sum_{j=3,4} b_j(y)\chi_Q(\Omega_j(y)), \\ \beta^2(y, Q) &= \int_{G_1 \cap Q} c(y, \eta)d\eta + \int_{\Gamma \cap Q} d(y, \eta)u(\eta)d\Gamma_{\eta} \end{aligned}$$

(for simplicity, we have omitted the subscript “1” in the notation of the measures). One can directly verify that these measures satisfy Conditions 2.2, 2.3, 3.2–3.5, and 4.1–4.3.

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References

- [1] J. M. Bony, P. Courrege, P. Priouret, “Semi-groups de Feller sur une variété à bord compacte et problèmes aux limites intégral-différentiels du second ordre donnant lieu au principe du maximum,” *Ann. Inst. Fourier* (Grenoble) **18**, (1968) 369–521.

- [2] W. Feller, “The parabolic differential equations and the associated semi-groups of transformations,” *Ann. of Math.* **55** 468–519 (1952).
- [3] W. Feller, “Diffusion processes in one dimension,” *Trans. Amer. Math. Soc.*, **77**, 1–30 (1954).
- [4] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1983.
- [5] E. I. Galakhov, A. L. Skubachevskii, “On contractive nonnegative semigroups with nonlocal conditions,” *Mat. Sb.*, **189**, 45–78 (1998); English transl.: *Math. Sb.* **189** (1998).
- [6] E. I. Galakhov, A. L. Skubachevskii, “On Feller semigroups generated by elliptic operators with integro-differential boundary conditions,” *J. of Differential Equations*, **176**, 315–355 (2001).
- [7] P. L. Gurevich, “Elliptic equations with nonlocal conditions near the conjugation points in the spaces of continuous functions,” *Tr. Mat. Inst. Steklova*; English transl.: *Proc. Steklov Inst. Math.*
- [8] Y. Ishikawa, “A remark on the existence of a diffusion process with non-local boundary conditions,” *J. Math. Soc. Japan*, **42**, 171–184 (1990).
- [9] K. Sato, T. Ueno, “Multi-dimensional diffusion and the Markov process on the boundary,” *J. Math. Kyoto Univ.* **4**, 529–605 (1965).
- [10] A. L. Skubachevskii, “On some problems for multidimensional diffusion processes,” *Dokl. Akad. Nauk SSSR*, **307**, 287–292 (1989); English transl. in *Soviet Math. Dokl.* **40** (1990).
- [11] A. L. Skubachevskii, “Nonlocal elliptic problems and multidimensional diffusion processes,” *Russian J. of Mathematical Physics*, **3**, 327–360 (1995).
- [12] K. Taira, *Diffusion Processes and Partial Differential Equations*, Academic Press, New York–London, 1988.
- [13] K. Taira *Semigroups, Boundary Value Problems and Markov Processes*. Springer-Verlag, Berlin 2004.
- [14] A. D. Ventsel, “On boundary conditions for multidimensional diffusion processes,” *Teor. Veroyatnost. i Primen.*, **4**, 172–185 (1959); English transl.: *Theory Probab. Appl.*, **4** (1959).